A Theoretical Framework for Sampling and Reconstructing Ensemble Average Propagators in Diffusion MRI

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Synopsis

Diffusion MRI can be modeled as sampling the Fourier transform of the Ensemble Average Propagator (EAP). This is potentially advantageous because of extensive theory that has been developed to characterize sampling requirements, accuracy, and stability for Fourier reconstruction. However, previous work has not taken advantage of this characterization. This work presents a novel theoretical framework that precisely describes the relationship between the estimated EAP and the true original EAP. The framework is applicable to arbitrary linear EAP estimation methods, and for example, provides new insights into the design of q-space sampling patterns and the selection of EAP estimation methods.

Purpose

Under the short-pulse approximation, the data measured in diffusion MRI can be modeled as the Fourier transform of the Ensemble Average Propagator (EAP) [1], a probability distribution that characterizes the molecular diffusion of the spins within each voxel. This is potentially advantageous because of extensive theory that has been developed to characterize sampling requirements, accuracy, and stability of Fourier reconstruction. However, existing diffusion MRI sampling and EAP estimation approaches have largely been developed and tuned without the benefit of such theory, instead relying on intuition, approximations, modeling assumptions, and extensive empirical evaluation. This work introduces a novel theory that can be used to characterize the performance of arbitrary linear EAP estimation methods with arbitrary q-space sampling schemes, building off of previous theoretical characterizations of orientation estimation [2-4]. For the first time, this provides a precise theoretical relationship between the true EAP and the estimated EAP. This relationship is similar to the point-spread function (PSF) relationship that is widely used to characterize the spatial resolution of MR images. In the context of EAP estimation, this theoretical framework provides direct insight into issues such as resolution, aliasing, and noise amplification. This valuable information can be directly used when choosing between different q-space sampling schemes and EAP estimation methods. Due to space constraints, we only describe the noiseless case in this abstract, though we also have a full noise-based theory.

Theory

Noiseless diffusion data $E(\mathbf{q}_m)$ measured at q-space location $(\mathbf{q}_m)$ can be modeled as the Fourier transform of the EAP, and a variety of linear and nonlinear EAP estimation methods have been designed for such data. This work focuses on linear methods like DSI [5], GQI [6], 3D-SHORE [7], SPF [8], SPFdual [9], SoH [10], and directional radial basis function estimation [11]. While each of these methods is based on different assumptions, they are all linear and can be characterized by the common reconstruction formula:

$$EAP_{estimated}(\mathbf{x}) = \sum_{m=1}^{M} E(\mathbf{q}_m) G(\mathbf{x}, \mathbf{q}_m)$$

where $G(\mathbf{x}, \mathbf{q}_m)$ are the method-dependent linear estimation coefficients. Due to the symmetry Fourier transform relationships, it is possible to expand this expression as

$$EAP_{estimated}(\mathbf{x}) = \int EAP_{true}(\boldsymbol{\xi}) g(\mathbf{x}, \boldsymbol{\xi}) d\boldsymbol{\xi}$$

where $g(\mathbf{x}, \boldsymbol{\xi}) = \sum_{m=1}^{M} G(\mathbf{x}, \mathbf{q}_m) cos(2\pi \mathbf{q}_m^T \boldsymbol{\xi})$ is the “EAP response function”. The EAP response function is similar to the standard PSF used in imaging. Similar to a PSF, the EAP response function should be similar to a delta function in order to achieve high fidelity EAP reconstruction, though some amount of blurring and aliasing is inevitable due to the finite sampling of q-space. Importantly, the EAP response function is easy to compute, and provides direct insight into the quality of a given q-space sampling-scheme and linear EAP estimation method.

Illustration

An illustration of the usefulness of our proposed theoretical framework is shown in Fig. 1, where we compare EAP responses for different q-space sampling schemes (multi-shell and Cartesian, each using 204 and 515 samples and maximum b-values of 4000 and 10000s/mm2) for GQI EAP estimation. As can be seen, the Cartesian sampling method suffers from coherent aliasing (seen as multiple peaks in the EAP response), while, multi-shell sampling scheme suffers from incoherent aliasing (side lobe interference). This expectation is confirmed with simulated data as shown in Fig. 2.

Note that there is a clear trade-off between the number of measured samples, the amount of aliasing interference (controlled by the sample spacing in q-space), and the resolution of the EAP (controlled by the size of the largest b-value). These effects are expected from traditional Fourier sampling theory, though have not usually been considered when designing diffusion experiments. As further illustration, Figs. 3 and 4 show that the EAP response can be used to theoretically predict the impact of user-selected reconstruction parameters. Specifically, we show EAP responses and estimated EAPs for 3D-SHORE EAP estimation using different values of the 3D-SHORE regularization parameter [7]. The EAP response behavior in Fig. 3 suggests that increasing regularization would initially improve accuracy by reducing sidelobe interference, though will eventually lead to a loss in EAP resolution if the regularization is increased too far. This theoretical prediction is confirmed by the estimation results shown in Fig. 4. This kind of characterization enables a new mechanism for optimizing reconstruction parameters.

Conclusion

We have proposed novel theoretical tools that can be used to characterize the performance linear diffusion MRI estimation methods without requiring extensive empirical testing. Notably, these tools do not require any assumptions about the EAP, and can be used even when traditional modeling assumptions are inaccurate. We expect these tools to be useful when designing sampling and estimation methods for a wide range of diffusion MRI experiments.

Acknowledgements

This work was supported in part by NSF CAREER award CCF-1350563 and NIH grant R01-NS089212.

References

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Figures

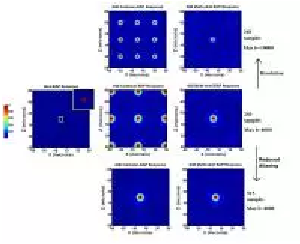

Fig.1:EAP response functions calculated for GQI reconstruction with (center) Cartesian and (right) multi-shell q-space sampling. We show results of varying the max b-value(top row) or the number of samples(bottom row) for both sampling schemes. EAP responses show the trade-off between resolution and aliasing artifacts as a function of these parameters.

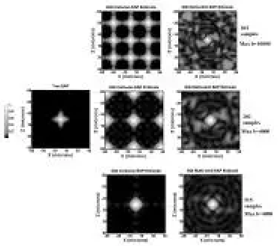

Fig.2:Estimated EAPs derived using GQI from (left) an ideal two-tensor crossing fiber simulation. The EAP estimation results for Cartesian and multi-shell sampling are consistent with the theoretical predictions from Fig. 1, as expected, confirming the usefulness of our novel theory for predicting the characteristics of different sampling and estimation schemes.

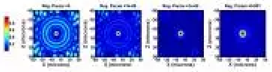

Fig.3:EAP response functions for 3D-SHORE, as a function of the 3D-SHORE regularization parameter. The EAP response has a narrower main lobe and smaller side lobes for intermediate values of the regularization parameter. Based on this, EAP estimation is expected to be poor for parameter values that are small or large.

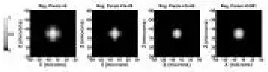

Fig.4:Estimated EAPs derived using 3D-SHORE from the ideal two-tensor crossing fiber simulation (shown in Fig. 2). The EAP estimation results for different regularization parameters are consistent with the theoretical predictions from Fig. 3, as expected.